TRANSVERSE VIBRATION ON A BEAM – AN ANALYTICAL MODEL

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Derivation of the wave equation

Consider a short section of a beam, length L and having cross-sectional area S. Due to a turning moment applied at x = L, the section is bent in an arc of radius R.

The moment at x = L may be derived by considering an element of the section, area dS, situated a distance k from the midline of the section. Due to the applied moment, the element is stretched a distance δx. The force dF required to stretch this element is

$$dF = \frac{E\, dS\, \delta x}{L}$$

where E is Young's modulus.

From symmetry, it can be seen that $\frac{\delta x}{k} = \frac{L}{R}$ so the equation for dF can be written

$$dF = \frac{E\, dS\, k}{R}$$

The moment about the midline is k dF, so the total moment M applied to the section at x = L will be

$$M = \int_S k\, dF = \frac{E}{R}\int_S k^2\, dS$$

$E\int_S k^2\, dS$ is the *flexural rigidity* of the beam and is usually written as EI where $I = \int_S k^2\, dS$ and is the *second moment of inertia* around the midline of the beam.

Let $SK^2 = \int_S k^2\, dS$ where K is the effective distance from the midline at which the total force F may be said to act.

Then the total moment is $M = \frac{ESK^2}{R}$

The radius R can be removed by expressing it in terms of x and y. $R^2 = x^2 + (R - y)^2$

Expanding, $R^2 = x^2 + R^2 - 2Ry + y^2$

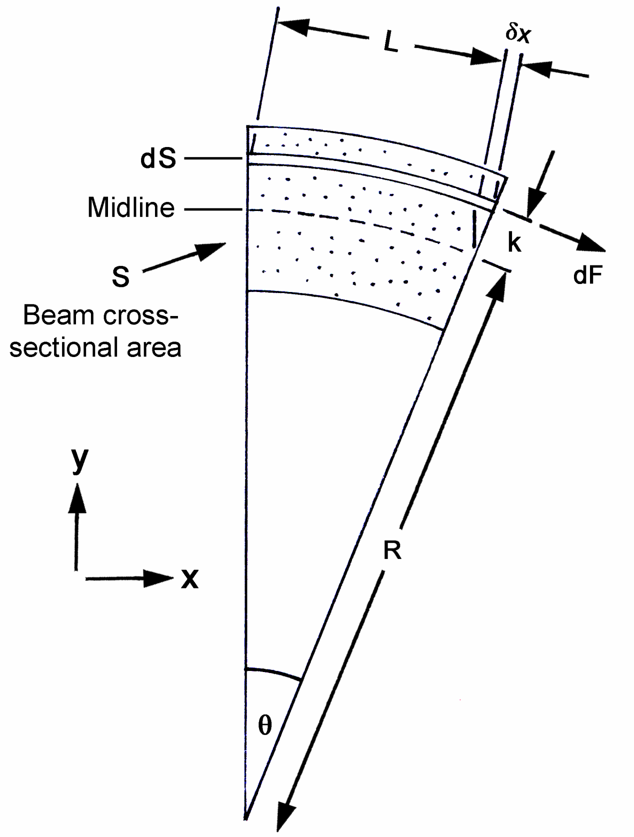

Figure 1

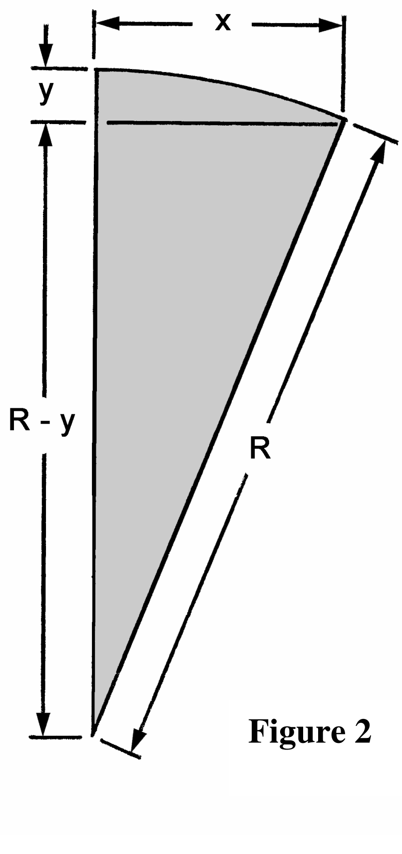

Figure 2

If x is small compared to R, y is always much smaller than x, so $y^2 \to 0$

and $y = \frac{x^2}{2R}$

Differentiating twice with respect to x, $\frac{d^2y}{dx^2} = \frac{1}{R}$

The total moment M can now be written as,

$$M = ESK^2 \frac{d^2y}{dx^2}$$

(Note that the second differential of y with respect to x is equivalent to the *total moment* at x .)

Let $F_y(x)$ be the function of the sheer force acting continuously along the length of the beam but orthogonally to it, such that the total sheer force acting on the beam is, $\int F_y(x)\, dx$

Now, the moment M will be equal and opposite to the sum of the sheer forces acting in the y direction, multiplied by their distance x from the left hand end of the segment.

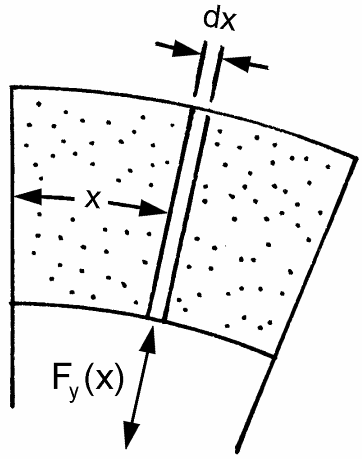

Figure 3

So, the total moment M for the segment can be expressed as, $\int_L x\, F_y(x)\, dx = -M = -ESK^2 \frac{d^2y}{dx^2}$

Differentiating both sides with respect to x gives, $x\ F_y(x) = -ESK^2 \frac{d^3y}{dx^3}$

(Note that the third differential of y with respect to x is equivalent to the *sheer force* on the beam over the length x. For a very short beam where $x \to dx$, this can be approximated to the sheer force *at* x.)

Differentiating once again, $F_y(x) + x\frac{dF_y(x)}{dx} = -ESK^2 \frac{d^4y}{dx^4}$

Now if the segment length L is very short, $F_y(x)$ can be considered a constant from $x = 0$ to $x = L$.

So $\frac{d}{dx}F_y(x) \to 0$ and then, $F_y(x) = -ESK^2 \frac{d^4y}{dx^4}$

The sheer force term $F_y(x)$ will be due to the transverse acceleration of beam at x, and can be written as

$F_y(x)\, dx = \rho\, S\, dx \frac{d^2y}{dt^2}$ where ρ is the density and $\rho\ S\ dx$ is the mass of the element through the segment at x of width dx.

Since $F_y(x)$ is a constant with respect to x over the length of the segment, it is possible to write $F_y(x) = \rho S \frac{d^2y}{dt^2}$ and finally, the equation of motion for transverse vibrations on a beam can be written as,

$$\frac{\partial^2 y}{\partial t^2} = -\frac{EK^2}{\rho}\frac{\partial^4 y}{\partial x^4}$$

Solutions to the wave equation

The partial differentiations in the wave equation above reflect the fact that y is an independent function of x and t such that; $y(x,t) = v(x)\,\text{Sin}\ \omega t$

Substituting this function into the wave equation gives, $v(x)\omega^2 = \frac{EK^2}{\rho}\frac{d^4 v(x)}{dx^4}$

Let $q^4 = \frac{\omega^2\rho}{EK^2}$ then the equation can be re-written as $q^4 v(x) = \frac{d^4 v(x)}{dx^4}$

The solutions to this equation are of the exponential type such that, $v(x) = c\,e^{\theta x}$ where c is an arbitrary constant.

It will seen that $\theta^4 = q^4$, so that $\theta^2 = \pm q^2$ and therefore θ can take on values of $\pm q$ or $\pm iq$ where $i = \sqrt{-1}$

The general solution then is, $v(x) = c_1 e^{qx} + c_2 e^{-qx} + c_3 e^{iqx} + c_4 e^{-iqx}$ where $c_{1....4}$ are arbitrary constants.

It is more convenient to express $v(x)$ in terms of trigonometric functions, so that the general solution of the x part of the wave equation can be expressed as;

$$v(x) = A\ \text{Cos}\ qx + B\ \text{Sin}\ qx + C\ \text{Cosh}\ qx + D\ \text{Sinh}\ qx$$

where A, B, C and D are arbitrary constants.

The particular solutions of this equation may be found for a given situation by subjecting it to boundary conditions.

The cantilever

Consideration of the solutions to the equations for transverse vibrations in the case of a fixed-free beam, the cantilever, serves as a model for the solutions for a free-free beam, pinned-free beam and beams with other boundary conditions applied.

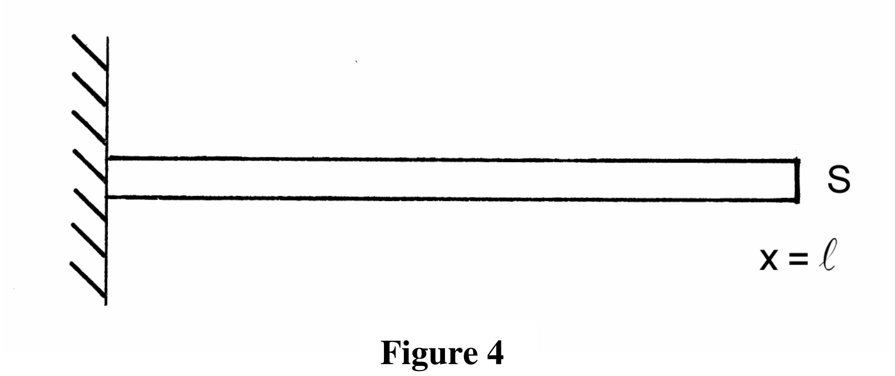

Figure 4

Figure 4 shows a beam, cross-sectional area S, length ℓ, which has one end fixed (x = 0) and the other end free (x = ℓ).

The fixed end is unable to move in the vertical direction, so v(x) = 0 at x = 0 for all time. Too, the angle of the beam does not change at x = 0, so $\frac{dv(x)}{dx} = 0$ at x = 0 for all time. Imposing these conditions on the general solution means that A + C = 0.

At the free end of the beam, there is necessarily no bending moment, so $\frac{d^2 v(x)}{dx^2} = 0$ at x = ℓ for all time.

And there will be no sheer force so $\frac{d^3 v(x)}{dx^3} = 0$ at x = ℓ for all time.

Imposing these conditions on the general solution means that B + D = 0.

So $C(\text{Cosh}\, q\ell + \text{Cos}\, q\ell) + D(\text{Sinh}\, q\ell + \text{Sin}\, q\ell) = 0$

and $C(\text{Sinh}\, q\ell - \text{Sin}\, q\ell) + D(\text{Cosh}\, q\ell + \text{Cos}\, q\ell) = 0$

To solve these homogeneous linear equations in C and D , it is necessary that;

$$\begin{vmatrix} \text{Cosh}\, q\ell + \text{Cos}\, q\ell & \text{Sinh}\, q\ell + \text{Sin}\, q\ell \\ \text{Sinh}\, q\ell - \text{Sin}\, q\ell & \text{Cosh}\, q\ell + \text{Cos}\, q\ell \end{vmatrix} = 0$$

or $1 + \text{Cosh}\, q\ell \; \text{Cos}\, q\ell = 0$

If $q\ell = Z$ then $\text{Cos}\, Z = -\frac{1}{\text{Cosh}\, Z}$ and this equation may be solved graphically by noting the values of Z at which the curves of these two functions intercept.

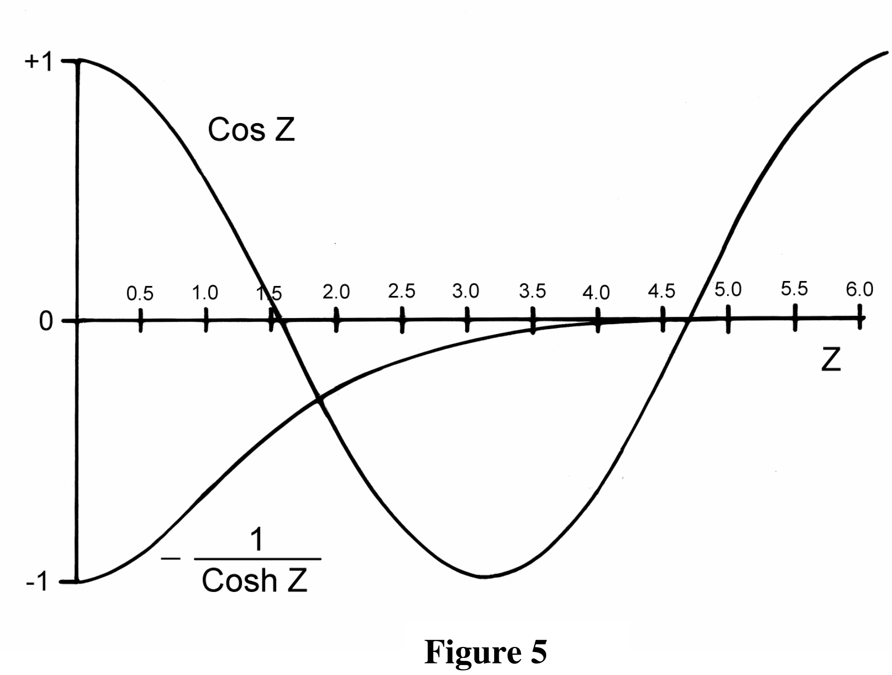

Figure 5

The first few roots are $Z_1 = 1.8751$ and $Z_2 = 4.694$. Thereafter, with the function $-\dfrac{1}{\text{Cosh}\, Z}$ approaching zero asymptotically, the roots are given to good accuracy by, $Z_n = (n+2)\dfrac{\pi}{2}$

The natural frequencies of a cantilever beam may now be written as,

$$\omega_n = q_n^2 \left(\frac{EK^2}{\rho}\right)^{\frac{1}{2}} = \left(\frac{Z_n}{\ell}\right)^2 \left(\frac{EK^2}{\rho}\right)^{\frac{1}{2}}$$

$$= \left(\frac{Z_n}{\ell}\right)^2 \left(\frac{EI}{\rho S}\right)^{\frac{1}{2}}$$

$$= \left(\frac{Z_n}{\ell}\right)^2 \left(\frac{EI}{m}\right)^{\frac{1}{2}}$$

where m is the mass per unit length of the beam.

The shape of the standing wave vibrational modes on the beam

As shown above, the amplitude $y(x)$ at x is given by;

$$v(x) = A \text{ Cos } qx + B \text{ Sin } qx + C \text{ Cosh } qx + D \text{ Sinh } qx$$

Since $A + C = 0$ and $B + D = 0$, then;

$$v(x) = C(\text{Cosh}\,qx - \text{Cos}\,qx) + D(\text{Sinh}\,qx - \text{Sin}\,qx)$$

using $C(\text{Sinh}\,q\ell - \text{Sin}\,q\ell) + D(\text{Cosh}\,q\ell + \text{Cos}\,q\ell) = 0$ to express D in terms of C, then dividing $v(x)$ through by C gives,

$$v(x) \equiv (\text{Cosh}\,qx - \text{Cos}\,qx) + \frac{(\text{Sin}\,q\ell - \text{Sinh}\,q\ell)}{(\text{Cosh}\,q\ell + \text{Cos}\,q\ell)}(\text{Sinh}\,qx - \text{Sin}\,qx)$$

Figure 6 shows the shapes of the first three modes of natural vibrations.

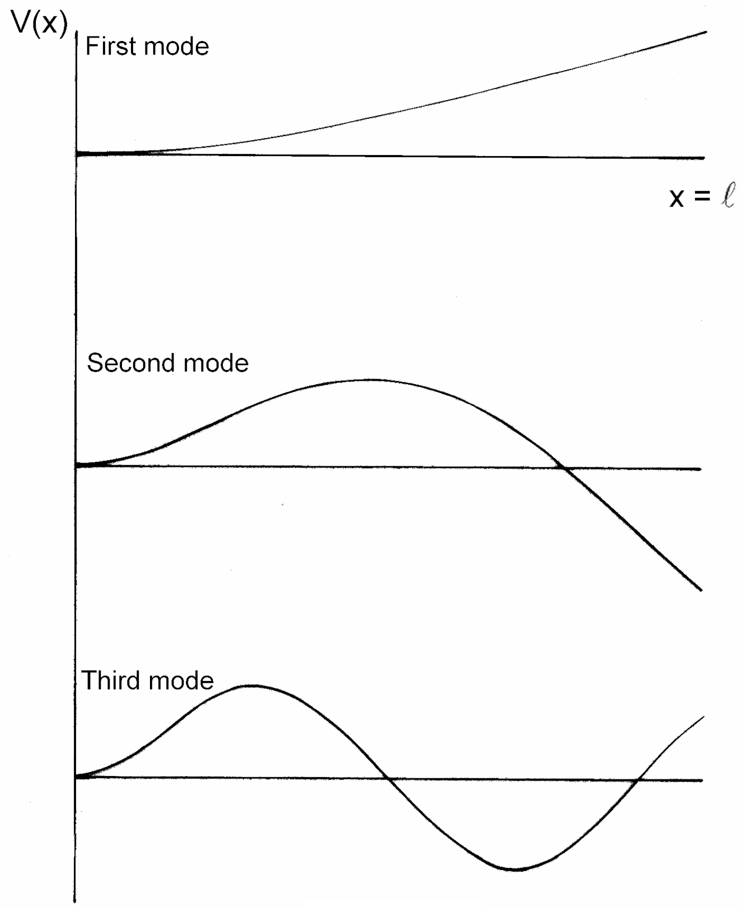

Figure 6

Transverse wave propagation velocity

While torsional vibrations, longitudinal vibrations and sheer vibrations will propagate along the beam at the speed of sound, $V_s = \sqrt{\frac{E}{\rho}}$, transverse vibration travels as a phase wave as given by the equation $y(x,t) = a\,\mathrm{Sin}\frac{2\pi}{\lambda}(x - Vt)$, which is a sinusoidal wave propagating along the beam.

This equation for y satisfies the wave equation $\frac{\partial^2 y}{\partial t^2} = -\frac{EK^2}{\rho}\frac{\partial^4 y}{\partial x^4}$ if $V = \frac{2\pi}{\lambda}\left(\frac{EK^2}{\rho}\right)^{\frac{1}{2}}$

It should be noted that the transverse vibration velocity is inversely proportional to the wavelength, λ and the velocity is thus dispersive with wavelength, or frequency.

In the trigonometric expressions above, the term $q\ell$ may be replaced by $q\ell = \frac{2\pi}{\lambda}\ell$ where λ is the wavelength of the particular resonant mode.

Substituting for λ, the expression for velocity may finally be written as, $V = \frac{Z}{\ell}\left(\frac{EK^2}{\rho}\right)^{\frac{1}{2}}$