# Simulating Transverse Vibrations in a Beam using a One Dimensional Lumped Parameter Finite Element Model

by

**Geoffrey Kolbe**
(Border Barrels Ltd., Newcastleton, Scottish Borders, TD9 0SN, UK)

*Introduction.*

When wishing to understand the behaviour of beams subject to an impulsive or complex, non-periodic moment applied to the ends, an analytic solution to the equations of motion is not possible and it is therefore necessary to resort to a numerical solution. Such a solution is normally attempted using a Finite Element simulation on a computer. However, commercially available FEA programs that are able to tackle time varying forces of a complex nature are usually aimed at the automobile or aircraft manufacturer, who need to calculate time dependent stress distributions and to simulate how their vehicles react in a crash. Consequently, programs of the complexity and power to render such scenarios are beyond the financial means and computing resources of individuals or small businesses.

However, instead of the generalised three dimensional mesh approach used in commercial FEA programs, it is probably sufficient to consider a beam as a one dimensional series of linked elements which affords great simplification both in the computer model and the computing power required to run it, without necessarily compromising the accuracy of the results.

A generalised lumped parameter model of a beam is described and an equation of motion is derived which is shown to be formally equivalent to the usual analytic equation of motion for transverse vibrations in a beam. It is then shown how this model may be implemented in a numerical model that can be easily and quickly programmed for a small computer or even a programmable calculator.

*The Lumped Parameter Model*

A simple mathematical model of a beam can be created by considering a lumped parameter system to represent a beam which is able to bend and vibrate in the vertical plane as shown in Figure 1. A light, flexible strip which is able to bend freely in the vertical plane represents the midline of the system. Attached to this strip are a set of discrete masses M. These masses are connected to a set of springs above and below the strip via light (but stiff) rods. Each pair of springs and the mass M to which they are connected by the rods may be considered as one element in a one dimensional series of N connected elements. A time varying moment applied at one end will be transmitted through the system via the springs and cause the masses to vibrate in the vertical plane.

One condition imposed on the system is that the light rods are connected at right angles (normally) to the flexible strip at all times.

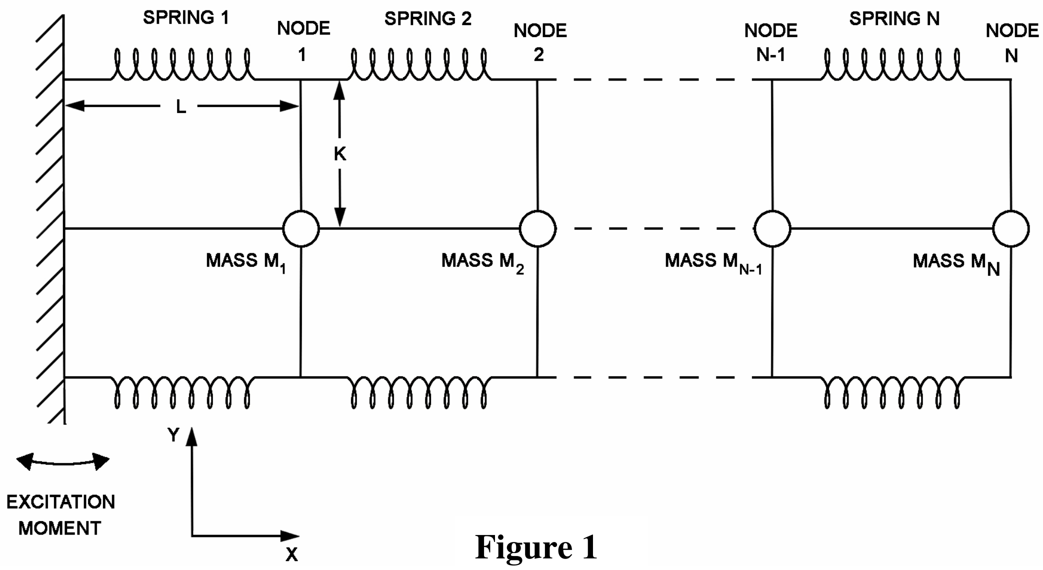


**Figure 1**

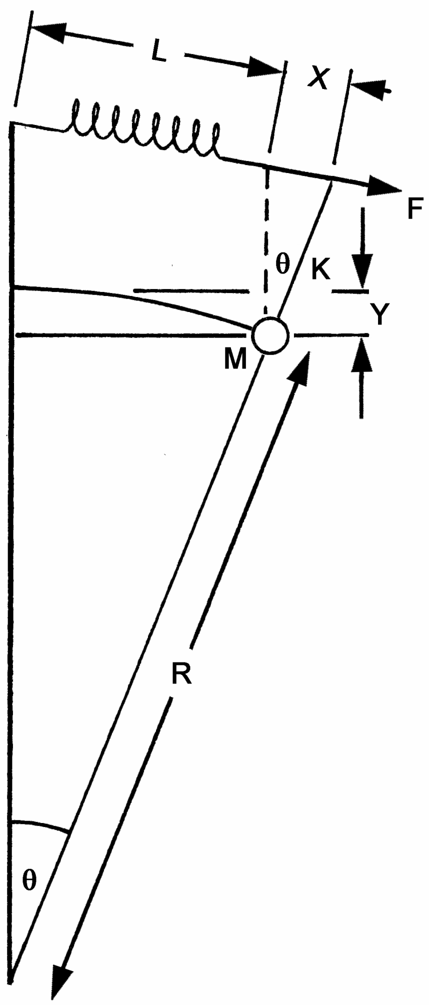


**Figure 2**

For the purpose of analysis, only one spring and rod associated with mass M need be considered as the spring system below the midline strip is a mirror image of that above it.

Let the length of each element be L and the height of the light rods be K. Let F be the force applied to a spring and let $F = \varphi X$, where $\varphi$ is the spring constant and X is the change in length of the spring due to the application of the force F.

As the spring changes in length, it causes the flexible strip to bend, displacing the mass in the vertical direction.

The flexible strip bends in an arc of radius R and subtending an angle $\theta$.

The displacement Y of the mass M can be written as,

$$Y = R - R\cos\theta$$

For small angles, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

Thus, $Y \approx R\frac{\theta^2}{2}$

Similarly, the spring extension X may be expressed as, $X = K\theta$ and by symmetry,

**Eqn. 1** $$\frac{X}{K} = \frac{L}{R}$$

Small changes $\Delta\theta$ in $\theta$ result in a change $\Delta Y$ such that, $\Delta Y = R\,\theta\,\Delta\theta = L\,\Delta\theta$

Also, from $\Delta\theta = \frac{\Delta X}{K}$

**Eqn. 2** $$\Delta Y = \frac{L}{K}\Delta X$$

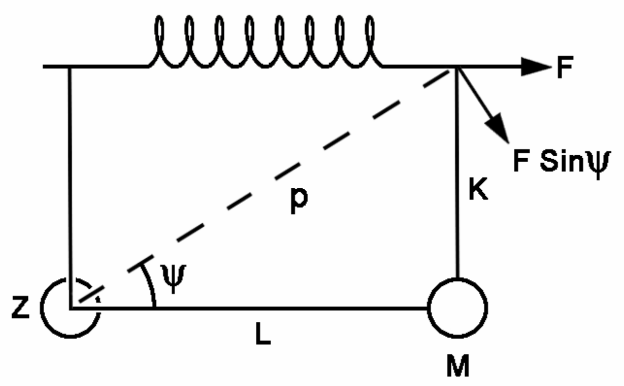


**Figure 3**

The acceleration of the mass M due to the force F may be obtained by considering the moments about Z in Figure 3.

The moments about Z are

$$2\,pF\,\mathrm{Sin}\psi = -L\,M\,\frac{d^2Y}{dt^2}$$

(The figure 2 on the left hand side accounts for the spring below the midline strip which is also acting on the mass M)

Now, $\mathrm{Sin}\psi = \frac{K}{\left(L^2+K^2\right)^{1/2}}$

and, $p = \left(L^2+K^2\right)^{1/2}$

So, the moments can be re-written as,

$$2FK = -L\,M\,\frac{d^2Y}{dt^2}$$

Substitute $F = \varphi X$ and the equation of motion for the mass M can be written in terms of the parameters $\varphi$ and K as,

**Eqn. 3** $$\frac{d^2Y}{dt^2} = -\frac{2\varphi K X}{M L}$$

*Quantifying the parameters for an actual beam*

The spring constant $\varphi$ can be written in terms of Young's modulus E such that $\varphi = \frac{ES}{2L}$

where S is the cross-sectional area of the beam. The factor 2 in the denominator is due to the fact that the system of springs depicted is mirrored above and below the midline. Thus the surface area to be considered associated to each spring in the model is half the cross-sectional area of the beam.

The mass M can be written in terms of the density $\rho$ of the beam such that $M = \rho L S$

Substituting for the parameters $\varphi$ and M in equation 2 yields the final equation of motion used in the computer model.

**Eqn. 4**
$$\frac{d^2Y}{dt^2} = -\frac{E\,K\,X}{\rho\,L^3}$$

In relation to an actual beam bent around its midline, as in Figure 4, the term K for the lumped parameter system may be considered as the effective distance at which the force F acts to apply a moment FK.

To determine K, consider the segment of beam in Figure 4, where an element of cross-sectional area dS is a distance k from the midline. The element is subject to a force dF and is extended a distance X. The force dF may be expressed as,

$$dF = \frac{E\,dS\,X}{L}$$

Using Eqn. 1 to rewrite dF,

$$dF = \frac{E\,dS\,k}{R}$$

The moment for this element about the midline is

$$k\,dF = \frac{E\,dS\,k^2}{R}$$

Integrating over half the cross-sectional area, the moment FK is then,

$$FK = \frac{E}{R}\int_{\frac{S}{2}} k^2\,dS$$

As an example, consider a tube with outer diameter 2b and inner diameter 2a, the moment FK can be expressed as,

$$FK = \frac{E}{R}\int_0^{\pi} d\alpha \int_a^b r^3 \,\mathrm{Sin}\alpha\, dr$$

where $k = r\,\mathrm{Sin}\alpha$ and $dS = r\,dr\,d\alpha$

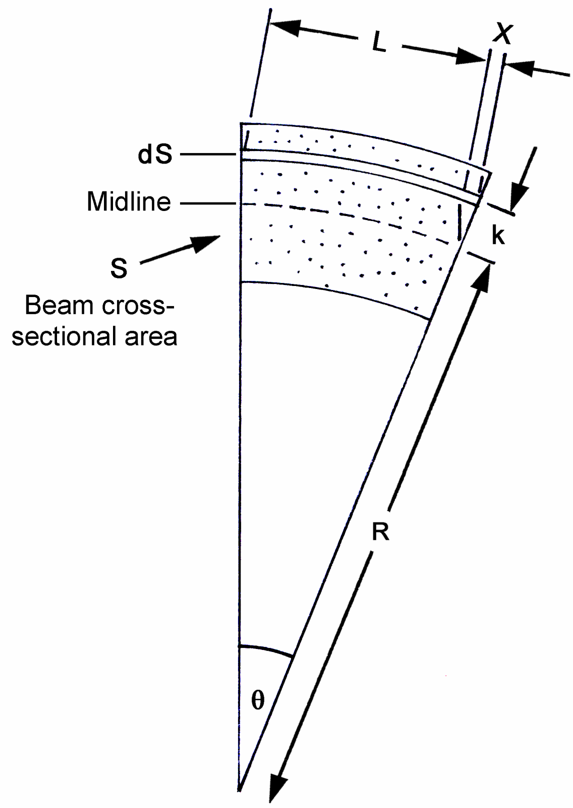


**Figure 4**

Integrating over r and α,

$$FK = \frac{\pi E\left(b^4 - a^4\right)}{8R}$$

Now, using Eqn.1, the force F on a spring can be written as

$$F = \varphi X = \frac{ESX}{2L} = \frac{ESK}{2R}$$

from which $R = \frac{ESK}{2F}$ and it follows that,

$$K^2 = \frac{\pi\left(b^4 - a^4\right)}{4S}$$

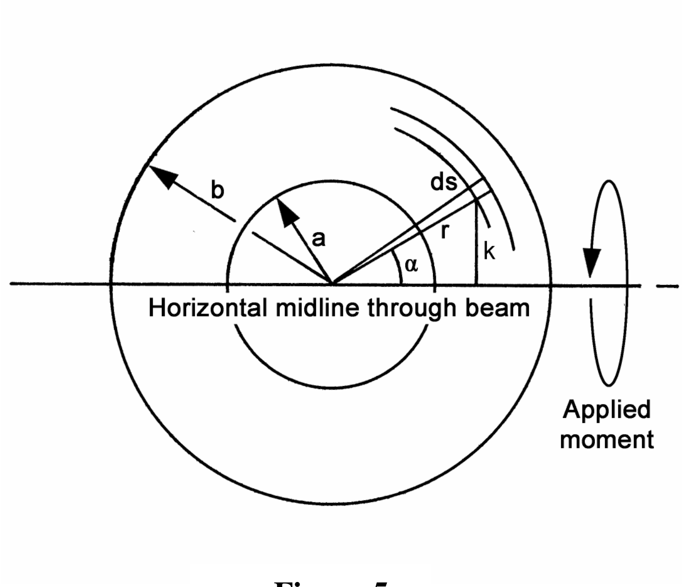


**Figure 5**

Since the cross-sectional area $S = \pi\left(b^2 - a^2\right)$ the parameter K for a tube can finally be expressed in terms of a and b as,

$$K = \frac{1}{2}\left(\frac{b^4 - a^4}{b^2 - a^2}\right)^{\frac{1}{2}}$$

*Validity of the lumped parameter model*

If the change X in the length of the spring L is small compared to L, then from Eqn. 2, the change ΔX in the length of the spring can be written as $\Delta X = \frac{K}{L}\Delta Y$

Substituting ΔX for X in Eqn. 4 reveals,

**Eqn. 5**

$$\frac{d^2Y}{dt^2} = -\frac{EK^2}{\rho}\frac{\Delta Y}{L^4}$$

This will be recognised as having the form of the usual analytic equation of motion for transverse vibrations on a beam. This shows that this lumped parameter approach as described is valid in treating the dynamics of a beam subject to external stresses.

*The computer model*

In this section, the way in which the lumped parameter model is implemented in a computer program is considered.

Let $Q = \frac{E K}{\rho L^3}$ then the equation of motion for a mass in the n$^{th}$ element $M_n$ becomes

$$\frac{d^2 Y_n}{dt^2} = Q X_n$$

For a single time step Δt, the following computations are executed sequentially for all N elements of the system.

Let $V_n(\text{start})$ be the velocity of the mass $M_n$ at the start of a time step Δt. The velocity at the end of the time step $V_n(\text{end})$ will be

$$V_n(\text{end}) = V_n(\text{start}) + \int_{\Delta t}\left(\frac{d^2 Y_n}{dt^2}\right)dt - C\int_{\Delta t} V_n(t)\,dt$$

The constant $C$ is a velocity dependent frictional damping term, necessary to ensure stability for the model. (The energy being put into the system has to go somewhere!). See the Appendix for an evaluation of this damping factor.

For a small time step Δt,

**Eqn. 6** $$V_n(\text{end}) \approx V_n(\text{start}) + Q\left[X_n(\text{start}) - X_{n+1}(\text{start})\right]\Delta t - C\,V_n(\text{start})\Delta t$$

Note that the net force present at the n$^{th}$ node will be a result of the extensions of n$^{th}$ spring and the n+1$^{th}$ spring.

The displacement $\Delta Y_n$ of the mass $M_n$ during the time step Δt will be

$$\Delta Y_n = \int_{\Delta t} V_n\,dt \approx \left[V_n(\text{start}) + V_n(\text{end})\right]\frac{\Delta t}{2}$$

The vertical $Y$ position of the mass $M_n$ at the end of the time step Δt will be

$$Y_n(\text{end}) = Y_n(\text{start}) + \Delta Y_n$$

The change $\Delta\theta_n$ in the angle $\theta_n$ of the n$^{th}$ rod to the vertical will be

$\Delta\theta_n = \frac{(\Delta Y_n - \Delta Y_{n-1})}{L}$ Note the $\Delta Y_{n-1}$ term. This is also how energy is propagated down the chain of elements.

Note that the change in angle of the n$^{th}$ rod depends on the difference in movements of the n$^{th}$ and the n-1$^{th}$ masses during the time step $\Delta t$. The change in angle of the n$^{th}$ rod is relative to the change in angle of the n-1$^{th}$ rod, to which the other end of the spring is attached.

The angle of the n$^{th}$ rod to the vertical at the end of the time step $\Delta t$ will be

$$\theta_n(\text{end}) = \theta_n(\text{start}) + \Delta\theta_n$$

Alternatively, the change $\Delta X_n$ in the extension $X_n$ of the n$^{th}$ spring during the time step $\Delta t$ will be

$\Delta X_n = \frac{K}{L}\left[\Delta Y_n - \Delta Y_{n-1}\right]$ Note the $\Delta Y_{n-1}$ term. The change in length of a spring depends on the movements of the masses at both ends of the spring. This is also how energy is propagated down the chain of elements.

The extension $X_n$ in the n$^{th}$ spring at the end of the time step $\Delta t$ will be

$$X_n(\text{end}) = K\left[\theta_n(\text{end}) - \theta_{n-1}(\text{end})\right]$$

or equivalently,

$$X_n(\text{end}) = X_n(\text{start}) - \Delta X_n + \Delta X_{n-1}$$

It now remains to reset the end values of the variables for all N elements to be the start values for the beginning of the next time step.

$$X_n(\text{start}) = X_n(\text{end})$$
$$V_n(\text{start}) = V_n(\text{end})$$
$$Y_n(\text{start}) = Y_n(\text{end})$$
$$\theta_n(\text{start}) = \theta_n(\text{end})$$

By storing the values $Y_n$(end) and $\theta_n$(end) as a function of time, it is possible to obtain the vertical displacements and angle to the vertical (or horizontal) of each element in the beam as a function of time.

*Appendix – Evaluation of the damping factor.*

The lumped parameter system described above may also be viewed as a series of strongly coupled oscillating systems, with a highly peaked "q" factor at the resonant frequency of the individual element, particularly if all the elements are the same size. In practice, this will be exhibited by a tendency for each mass to oscillate up and down at the resonant frequency, each in anti-phase with its neighbouring mass. To prevent this, each element should be critically damped at the resonant frequency.

The undamped resonant frequency of each element may be derived as follows:

Noting from above that; $Y \approx R\frac{\theta^2}{2}$ and $X = K\,\theta$ then substituting into Eqn. 1, $X$ may be expressed in terms of $Y$ as, $X = \frac{2KY}{L}$ Substituting into Eqn. 4 reveals,

**Eqn. 7** $$\frac{d^2Y}{dt^2} = -\frac{2E\,K^2\,Y}{\rho\,L^4} = -\omega^2 Y$$

where $\omega$ is the resonant frequency of each element such that,

**Eqn. 8** $$\omega_r = \sqrt{2}\left(\frac{E\,K^2}{\rho\,L^4}\right)^{\frac{1}{2}}$$

The period of the resonant frequency will be $\tau_r$ where, $\tau_r = \frac{2\pi}{\sqrt{2}}\left(\frac{\rho\,L^4}{EK^2}\right)^{\frac{1}{2}}$

Starting with Eqn. 6 above, which describes the vertical velocity of the mass in each element,

$$V_n(\text{end}) \approx V_n(\text{start}) + Q\left[X_n(\text{start}) - X_{n+1}(\text{start})\right]\Delta t - C\,V_n(\text{start})\Delta t$$

This may be re-expressed as, $\frac{\Delta V_n}{\Delta t} = Q\,\Delta X_n - C\,V_n(\text{start})$

Now, if the element is critically damped, the acceleration of the mass will be zero and so,

$Q\,\Delta X_n = C\,V_n(\text{start})$ Then, substituting Eqn. 2, and re-writing, $\frac{E\,K^2\,\Delta Y_n}{\rho\,L^4} = C\frac{\Delta Y_n}{\Delta t}$

or, $C = \frac{E\,K^2\Delta t}{\rho\,L^4}$

For critical damping at the resonant frequency, the term $\Delta t$ in the equation for $C$ will be related to $\tau_r$ the period of the resonant frequency. From information theory, to sample a frequency of period $\tau_r$, the period $\Delta t$ needs to be set to 0.5 $\tau_r$

The final expression for the damping factor $C$ is then, $C = \frac{\pi}{\sqrt{2}}\left(\frac{E\,K^2}{\rho\,L^4}\right)^{\frac{1}{2}} \approx 2.22\left(\frac{E\,K^2}{\rho\,L^4}\right)^{\frac{1}{2}}$

In fact, a value of $C \approx 1.7\left(\frac{E K^2}{\rho L^4}\right)^{\frac{1}{2}}$ would appear to be a more suitable value in practice, and the model is not well behaved if the value of $C$ varies from this value by more than ±10%

*Conclusions*

The model described here can be applied relatively easily to actual physical situation. The model appears to be reasonably easy to implement and run-time is fast. Benchmark tests to simulate a cantilever (fixed-free beam) produced a first mode frequency vibration that was within 3% of the theoretical value.

The critical damping frequency is clearly the upper limit of the frequency response of the model and the effect of the damping factor is to apply a low pass filter with a roll-off of 3db per octave from the resonant frequency. Of course, the shorter the elements are, the higher the frequency response of the model will be. It is seen from Eqn. 8 that the element resonant frequency goes as the inverse of the element length squared, so significant improvements in frequency response are achieved by shortening the elements. But there is a limit as the model becomes unstable if the element length is set significantly shorter than the element width. Too, if the elements are shortened, there needs to be an increasing number of them to model the length of the beam and so the computing time will be lengthened.